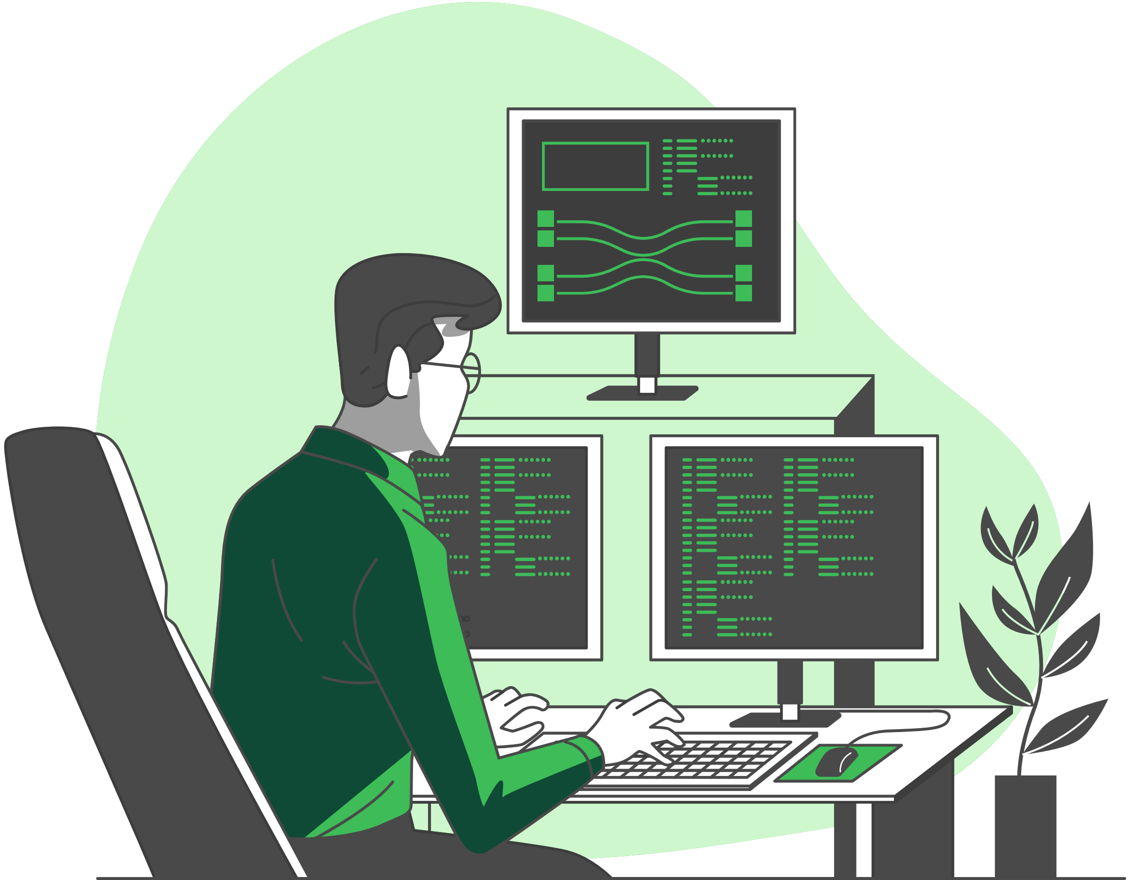

Transitive Closure

CS 251 - Data Structures and Algorithms

Note:
Slides complement the discussion in class

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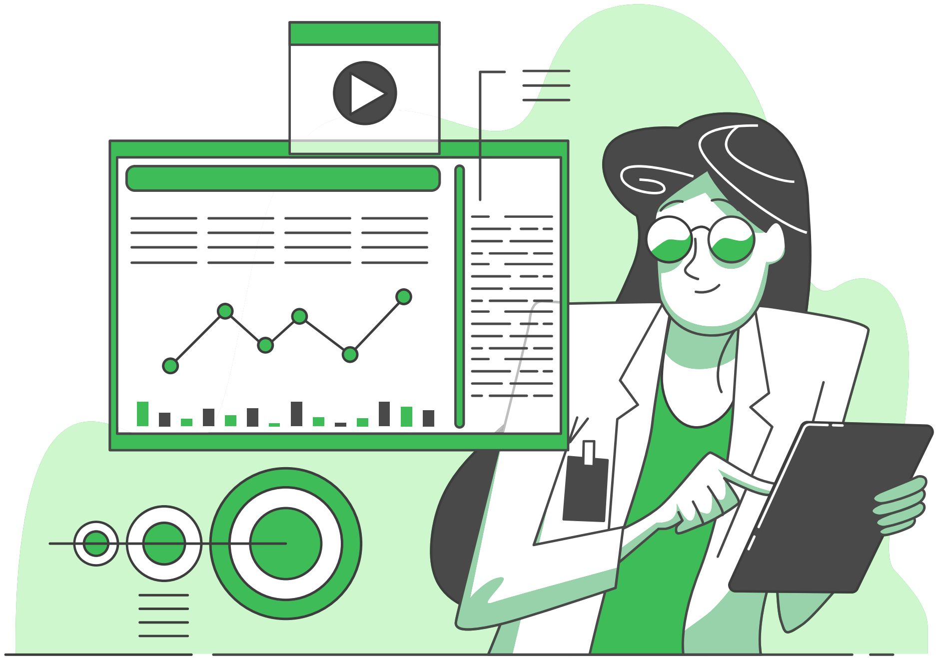


01

Transitive Closure

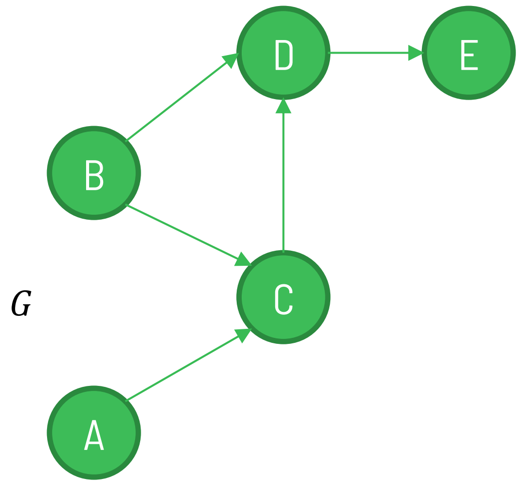
Information about reachability

Transitive Closure

Given a graph G, the transitive closure of G is the digraph G^* such that:

- G^* has the same vertices as G.
- If G has a directed path from u to v (with $u \neq v$), G^* has a directed edge from u to v.

The transitive closure provides reachability information about a digraph.



Transitive Closure

Given a graph G, the transitive closure of G is the digraph G^* such that:

- G^* has the same vertices as G.
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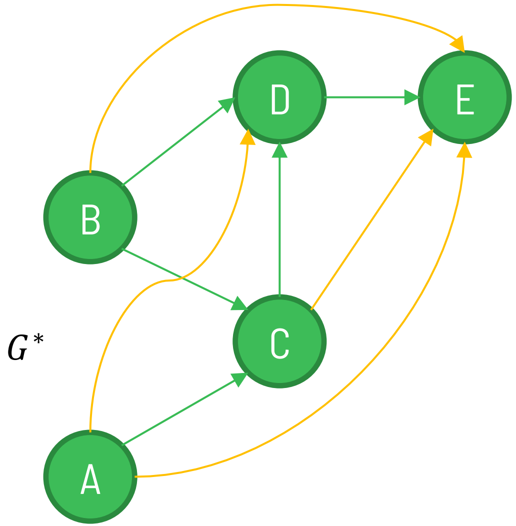
The transitive closure provides reachability information about a digraph.

Note: Some transitive closure definitions allow $(u = v)$.



Transitive Closure Variations

1. **(Sedgewick and Wayne):** On the adjacency matrix, automatically add 1's down the diagonal because every vertex is reachable from itself (i.e., every vertex has a self-loop in the transitive closure).

2. **(Goodrich and Tamassia):** On the adjacency matrix, set the diagonal to all 0's unless a self-loop exists in the original graph.

3. **(Warshall):** On the adjacency matrix, the transitive closure will have 1's in the diagonal if:
   a. There is a self-loop in the original graph.
   b. There is a cycle in the original graph containing that vertex

Floyd–Warshall Transitive Closure

algorithm Floyd-Warshall(M:adjacency matrix representing $G(V,E)$)

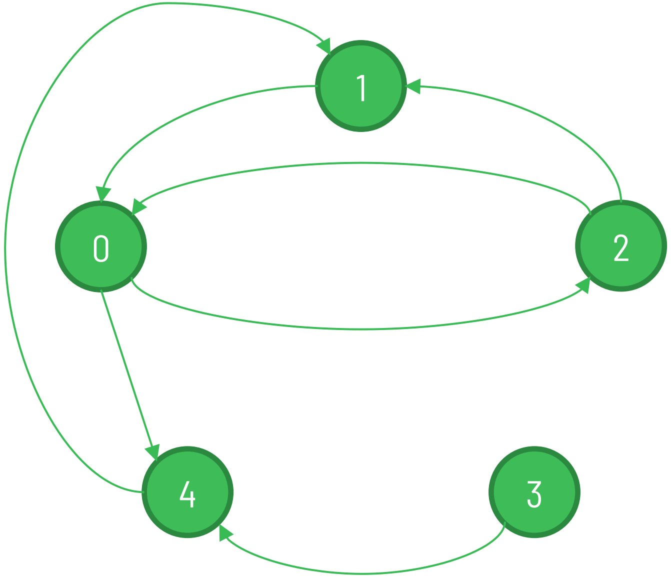
$R^{(-1)} \leftarrow M$
$n \leftarrow |V|$

for k from 0 to $n-1$ **do**
  for i from 0 to $n-1$ **do**
    for j from 0 to $n-1$ **do**
      $R^{(k)}[i,j] \leftarrow R^{(k-1)}[i,j]$ or $(R^{(k-1)}[i,k]$ and $R^{(k-1)}[k,j])$
    end for
  end for
end for

return $R^{(n-1)}$
end algorithm

Floyd-Warshall:

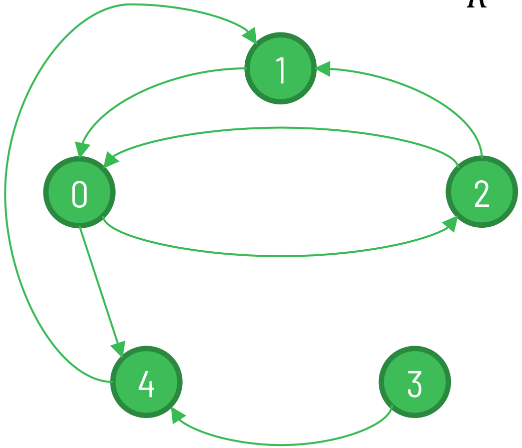
0:	2, 4
1:	0
2:	0, 1
3:	4
4:	1



$R^{(-1)}$

	0	**1**	**2**	**3**	**4**
0	-	-	1	-	1
1	1	-	-	-	-
2	1	1	-	-	-
3	-	-	-	-	1
4	-	1	-	-	-

Remember: $R^{(k)}[i,j] \leftarrow R^{(k-1)}[i,j]$ or $(R^{(k-1)}[i,k]$ and $R^{(k-1)}[k,j])$



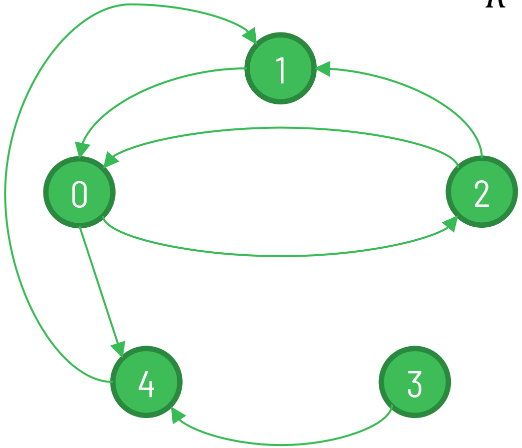
$R^{(-1)}$

	0	**1**	**2**	**3**	**4**
0	-	-	1	-	1
1	1	-	-	-	-
2	1	1	-	-	-
3	-	-	-	-	1
4	-	1	-	-	-

$R^{(0)}$

	0	**1**	**2**	**3**	**4**
0	-	-	1	-	1
1	1	-	1	-	1
2	1	1	1	-	1
3	-	-	-	-	1
4	-	1	-	-	-

Remember: $R^{(k)}[i,j] \leftarrow R^{(k-1)}[i,j]$ or $(R^{(k-1)}[i,k]$ and $R^{(k-1)}[k,j])$



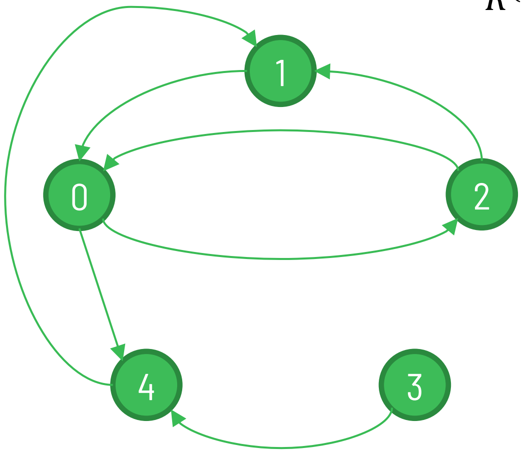
$R^{(0)}$

	0	**1**	**2**	**3**	**4**
0	-	-	1	-	1
1	1	-	1	-	1
2	1	1	1	-	1
3	-	-	-	-	1
4	-	1	-	-	-

$R^{(1)}$

	0	**1**	**2**	**3**	**4**
0	-	-	1	-	1
1	1	-	1	-	1
2	1	1	1	-	1
3	-	-	-	-	1
4	1	1	1	-	1

Remember: $R^{(k)}[i,j] \leftarrow R^{(k-1)}[i,j]$ or $(R^{(k-1)}[i,k]$ and $R^{(k-1)}[k,j])$



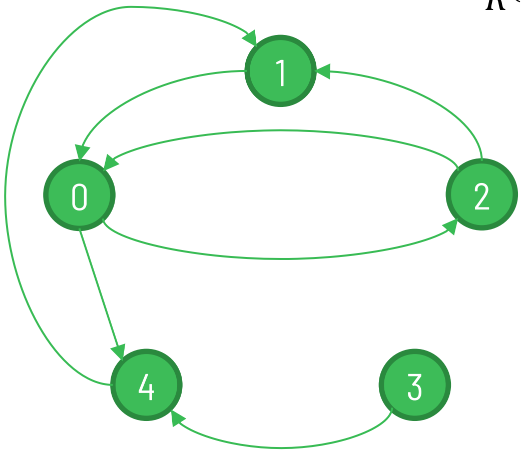
$R^{(1)}$

	0	1	2	3	4
0	-	-	1	-	1
1	1	-	1	-	1
2	1	1	1	-	1
3	-	-	-	-	1
4	1	1	1	-	1

$R^{(2)}$

	0	1	2	3	4
0	1	1	1	-	1
1	1	1	1	-	1
2	1	1	1	-	1
3	-	-	-	-	1
4	1	1	1	-	1

Remember: $R^{(k)}[i,j] \leftarrow R^{(k-1)}[i,j]$ or $(R^{(k-1)}[i,k]$ and $R^{(k-1)}[k,j])$



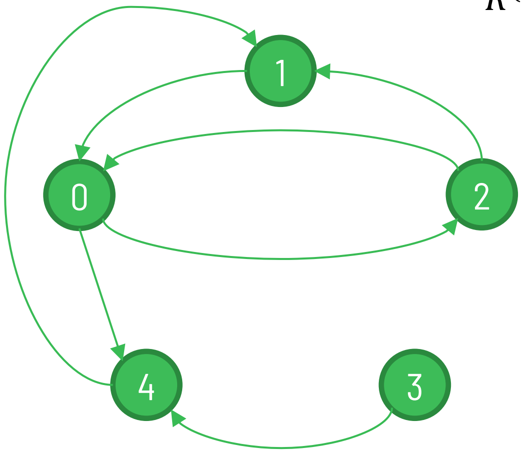
$R^{(2)}$

	0	1	2	3	4
0	1	1	1	-	1
1	1	1	1	-	1
2	1	1	1	-	1
3	-	-	-	-	1
4	1	1	1	-	1

$R^{(3)}$

	0	1	2	3	4
0	1	1	1	-	1
1	1	1	1	-	1
2	1	1	1	-	1
3	-	-	-	-	1
4	1	1	1	-	1

Remember: $R^{(k)}[i,j] \leftarrow R^{(k-1)}[i,j]$ or $(R^{(k-1)}[i,k]$ and $R^{(k-1)}[k,j])$



$R^{(3)}$

	0	1	2	3	4
0	1	1	1	-	1
1	1	1	1	-	1
2	1	1	1	-	1
3	-	-	-	-	1
4	1	1	1	-	1

$R^{(4)}$

	0	1	2	3	4
0	1	1	1	-	1
1	1	1	1	-	1
2	1	1	1	-	1
3	1	1	1	-	1
4	1	1	1	-	1

Floyd–Warshall Algorithm Discussion

Q: Assuming an adjacency matrix representation, what is the space requirement for the algorithm?
A: $O(|V|^2)$

Q: What is the runtime of the algorithm?
A: $O(|V|^3)$

Q: Assuming no self-loops in the original graph, what does a self-loop (a 1 in the diagonal of the matrix) in the transitive closure graph tell us?
A: There is a cycle in the graph containing that vertex.

Q: What does it mean if the resulting matrix is all 1's?
A: Every vertex is reachable from every other vertex.

Question

0	1	1	1	1
1	0	1	1	1
1	1	0	1	1
1	1	1	0	1
1	1	1	1	0

Q: Is this matrix a possible transitive closure of a directed graph?

A: Not with Warshall's variation. If every vertex is reachable from every other vertex, then there should be 1's down the diagonal since every vertex is part of a cycle.

Question

1	0	0	0	0
0	1	0	0	0
0	0	1	0	0
0	0	0	1	0
0	0	0	0	1

Q: Is this matrix a possible transitive closure of a directed graph?

A: Yes (with Warshall's variation). If the original graph is 5 unconnected vertices, each with a self-loop.

We’re Done

Do you have any questions?

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